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Technical Note

Duhamel integral form for the interface heat flux between bubble and liquid

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Abstract

The solution of heat equation inside oscillating gas bubble with moving boundary was obtained by Fourier's method. The integral formula for interface heat flux, containing theta-function in the integrand was derived. The kernel of the integral is represented by a series of exponential functions, and a simple analytic approximation obtained earlier is used for it with high accuracy. The asymptotic expression for the interface heat flux in the Duhamel integral form with rooted kernel was derived.

The vapor bubbles were also considered. In this case the major problem is external heat problem in liquid. It is shown that the asymptotic expression for the heat flux at the interface in the case of gas bubbles has the similar structure as the heat flux from the vapor bubble surface to the liquid. In both cases it is Duhamel integral with rooted kernel.

Keywords: Heat flux; Duhamel integral; Gas bubble; Vapor bubble

1. Introduction

Mass, force, and energy interactions proceeding at the interfaces in gas–liquid flows result in substantial variations in the flow velocity, pressure, and temperature field. To describe the processes of interface heat-exchange in gas–liquid bubble flows correctly, one has to analyze the interaction between a bubble and the carrier phase.

Oscillations of gas and vapor bubbles in liquid have been analyzed, both theoretically and experimentally, in a number of works which are discussed in detail in [1–3]. However, of both methodological and practical importance is the derivation of simple analytical dependencies for interface heat-exchange.

2. Formulation of the problem for the gas bubble

Let us consider the behavior of a gas bubble in a liquid. The surrounding liquid is incompressible and ideal. The processes occurring within a bubble are assumed to be spherically symmetric. Phase transitions are ignored due to the low temperature of the gas and the liquid.

The pressure inside the bubble is assumed to be uniform [4–6]. It takes place when the length of sound wave in gas is much greater than the bubble radius. In the absence of phase transitions the temperature of a liquid remains practically unchanged, and the heat flux across the interface, q, is fully defined by the thermal resistance of a gas. Since no phase transitions take place, the heat flux across the boundary is continuous. Hence, q may be found by solving the internal heat-exchange problem for a bubble [4–7]

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial r}=\frac{a}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\gamma-1}{\gamma}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{2.1}$$

$$T(r,0)=T_0, T(R,t)=T_0, \frac{\partial T}{\partial r}=0(r=0), p(0)=p_0 \tag{2.2}$$

The temperature of the bubble surface stays practically constant since the liquid has much higher thermal conductivity and much smaller thermal diffusivity than the gas [8].

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Nomenclature

R	bubble radius
$\dot{R}$	time derivative of the radius
r	radial Euler coordinate
x	longitudinal coordinate
t	time
j	the phase transition rate
ℓ	the latent heat of evaporation
$Pe = 2R_0(3\gamma p_0/\rho_e)^{1/2}a_g$	Peclet number
T	temperature
z	dummy variable (nondimensional)
ρ	density
p	pressure
v	radial velocity
V	longitudinal velocity
λ	thermal conductivity
a	thermal diffusivity
B	gas constant
γ	specific heat ratio
C	specific heat
C_p	specific heat of the gas at constant pressure
$P = \frac{p}{p_0}, \xi = \frac{r}{R}, \theta = \frac{T}{T_0}, \tau = \frac{t}{t^*}$	nondimensional parameters
$t_* = \frac{R_0^2}{a_g}, u = \theta\xi, \tau_0 = x_1/\pi^2 = 0.1548, x_1 = 1.526$	

Subscripts

l	liquid
g	gas
v	vapor
O	at equilibrium
R	at the bubble surface
S	at the saturation

The continuity equation for the gas, with using the uniformity of the pressure and the boundary condition $v(0, t) = 0$, yields the velocity profile in the bubble:

$$v(r,t) = \frac{r}{R}\dot{R} + \frac{\gamma-1}{\gamma}\left[\lambda\frac{\partial T}{\partial r} - \frac{r}{R}\left(\lambda\frac{\partial T}{\partial r}\right)_{\mathrm{R}}\right] \tag{2.3}$$

3. Analytic solution

The problem was solved by using the variable $\xi = r/R(t)$ which "freezes" the moving boundary of the bubble. By using formulas for the change of variables:

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_r &= \left.\frac{\partial}{\partial t}\right|_\xi + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} = \left.\frac{\partial}{\partial t}\right|_\xi - \frac{r}{R^2}\dot{R} \\ \frac{\partial}{\partial r} &= \frac{1}{R(t)}\frac{\partial}{\partial \xi} \end{aligned} \tag{3.1}$$

we obtain

$$\frac{\partial T}{\partial t} + \left(\frac{v - \xi\dot{R}}{R}\right)\frac{\partial T}{\partial \xi} = \frac{a}{R^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial T}{\partial \xi}\right) + \frac{\gamma-1}{\gamma}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{3.2}$$

The estimates show that deviation of velocity profile inside the bubble from linear dependence due to the temperature gradient (2.3) is usually not exceed 20%. For this reason the convective term of heat Eq. (3.2) in new variables (ξ,t) can be neglected.

Let us consider the case when the deviation of bubble's radius from equilibrium position is small enough. Then Eq. (3.2) can be simplified

$$\frac{\partial T}{\partial t} = \frac{a}{R_0^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial T}{\partial \xi}\right) + \frac{\gamma-1}{\gamma}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{3.3}$$

Let us use nondimensional variables

$$P = \frac{p}{p_0}, \theta = \frac{T}{T_0}, \tau = \frac{t}{t_*}, t_* = \frac{R_0^2}{a_{\mathrm{g}}} \tag{3.4}$$

Eq. (3.3) in nondimensional variables will have a form:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial \theta}{\partial \xi}\right) + \frac{\gamma-1}{\gamma}\frac{\mathrm{d}(\ln P)}{\mathrm{d}\tau}\theta \tag{3.5}$$

$$\theta(\xi,0) = \theta_0, \theta(1,\tau) = \theta_0, \frac{\partial \theta}{\partial \xi}(0,\tau) = 0$$

$$P(0) = 1$$

Let us use a new variable $u = \theta\xi$. In new variables Eq. (3.5) will have a form:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\gamma-1}{\gamma}\frac{\mathrm{d}(\ln P)}{\mathrm{d}\tau}u \tag{3.6}$$

Solving this nonhomogeneous equation by Fourier's method we will obtain the solution

$$\begin{aligned} u &= \xi\theta_0 + \sum_{n=1}^{\infty} a_n(\tau)\sin \pi n\xi \\ a_n(\tau) &= \int_0^\tau q_n(z)\exp[-(\pi n)^2(\tau - z)]\mathrm{d}z \\ q_n(z) &= 2\frac{\gamma-1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}z}\int_0^1 \xi\sin(\pi n\xi)\mathrm{d}\xi \\ &= 2\frac{\gamma-1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}z}\frac{(-1)^{n+1}}{\pi n} \end{aligned} \tag{3.7}$$

$$u = \xi\theta_0 + \frac{2(\gamma-1)}{\gamma}\sum_{n=1}^{\infty}\int_0^\tau \frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\frac{(-1)^{n+1}}{\pi n} \times \exp[-(\pi n)^2(\tau - z)]\mathrm{d}z\sin(\pi n\xi)$$

$$\theta = \theta_0 + \frac{2(\gamma-1)}{\gamma}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi n}\int_0^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}$$
$$\times \exp[-(\pi n)^2(\tau - z)]\mathrm{d}z\frac{\sin(\pi n\xi)}{\xi}$$

From temperature distribution (3.7) we can find heat flux at the interface

$$q = -\lambda_g \frac{\partial T}{\partial r}\bigg|_{r=R} = -\lambda_g \frac{T_0}{R_0}\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=1}$$
$$= \frac{2(\gamma-1)}{\gamma}\frac{\lambda_g T_0}{R_0}\int_0^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\sum_{n=1}^{\infty}\exp[-(\pi n)^2(\tau - z)]\mathrm{d}z \tag{3.8}$$

The kernel of integral (3.8) can be written as follows:

$$G(x) = 2\sum_{n=1}^{\infty}\exp(-n^2 x) = \psi(x) - 1$$
$$\psi(x) = \sum_{n=-\infty}^{\infty}\exp(-n^2 x) \tag{3.9}$$

The function $\psi(x)$ can be expressed in terms of the Jacobi theta-function. The following high accuracy approximation for the function $G(x)$ was obtained in [9]

$$G(x) = \begin{cases} \sqrt{\pi/x} - 1, 0 < x \leqslant x_1 \\ 2\exp(-x), x \geqslant x_1 \end{cases} \tag{3.10}$$

It was obtained in [9] that $x_1 = 1.526$ and relative error of approximation (3.10) amounts to 1%. In Fig. 1, the plot of the exact values of the function $G(x)$ is presented. One can readily verify that the approximation (3.10) of the function $G(x)$ for $x \leqslant x_1$, and $x > x_1$ practically coincide with the exact curve given in Fig. 1.

Expression (3.8) taking (3.9) into account can be written in the form:

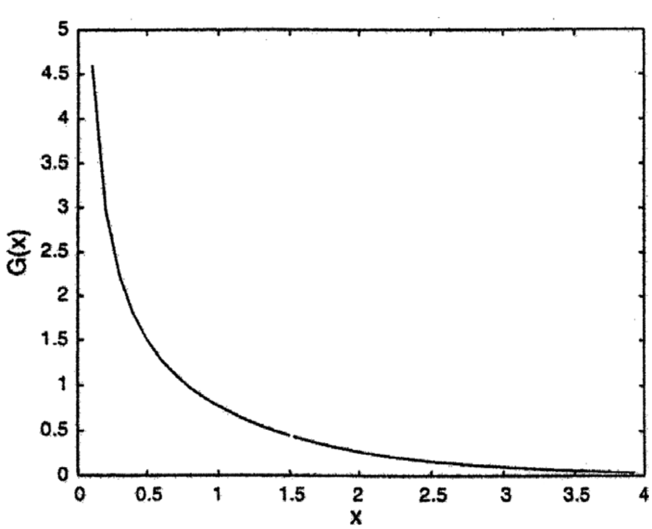

Fig. 1. Plot of the exact values of $G(x)$.

$$q = \frac{\gamma-1}{\gamma}\frac{\lambda_g T_0}{R_0}\int_0^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}G[\pi^2(\tau - z)]\mathrm{d}z \tag{3.11}$$

Taking into account (3.10) we will obtain:

$$q = \frac{\gamma-1}{\gamma}\frac{\lambda_g T_0}{R_0}\left\{\int_0^{\tau-\tau_0} 2\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\exp[-\pi^2(\tau - z)]\mathrm{d}z\right.$$
$$\left. + \int_{\tau-\tau_0}^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\frac{\mathrm{d}z}{\sqrt{\pi(\tau - z)}} - \int_{\tau-\tau_0}^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\mathrm{d}z\right\} \tag{3.12}$$

$\tau > \tau_0 = x_1/\pi^2 = 0.1548$

Let us introduce Peclet number

$$Pe = \frac{2R_0}{a_g}\left(\frac{3\gamma p_0}{\rho_e}\right)^{1/2}$$

As it is shown in [9] for $Pe \ll 1$ (in the case of bubble oscillations, close to isothermal) it is necessary to take into account in (3.12) all three terms in the braces.

But in another asymptotic case $Pe \gg 1$ (for the oscillations of a bubble close to adiabatic) it can be shown that the major term in (3.12) is the middle term.

Hence

$$q(\tau) = \frac{\gamma-1}{\gamma}\frac{\lambda_g T_0}{R_0\sqrt{\pi}}\int_{\tau-\tau_0}^{\tau}\frac{\mathrm{d}(\ln P)}{\mathrm{d}z}\frac{\mathrm{d}z}{\sqrt{\tau - z}} \tag{3.13}$$
$$\tau > \tau_0, Pe \gg 1$$

4. Vapor bubbles

In the case of vapor bubble in hot liquid when the phase transitions take place the major role plays the external heat problem in liquid surrounding the bubble [4].

The boundary condition at the interface have a form

$$r = R : j\ell = q_e - q_v, q_e = -\lambda_e\frac{\partial T_e}{\partial r}, q_v = -\lambda_v\frac{\partial T_v}{\partial r} \tag{4.1}$$

usually $|q_v| \ll |q_e|$.

We assume that the vapor obeys the equation of state of a perfect gas, and being in the saturated state at the interface it obeys the Clapeyron–Clausius equation

$$\frac{\mathrm{d}T_s}{\mathrm{d}p_v} = \frac{T_s(p)}{\ell\rho_v}, \rho_v \ll \rho_l \tag{4.2}$$

As the first approximation for q_e (the heat flux from the bubble surface to liquid), we can use solution [10].

This solution assumes that the bubble boundary is not moving and boundary conditions are nonsteady.

$$q_e = \lambda_e\frac{T_v(0) - T_0}{\sqrt{\pi a_e t}} + \lambda_e\frac{T_v(t) - T_0}{R} + \frac{\lambda_e}{\sqrt{\pi a_e}}\int_0^t\frac{\mathrm{d}T_s(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\sqrt{t-\tau}} \tag{4.3}$$

First term in (4.3) express the contribution to the heat flux of the sudden (jump) change of boundary condition. The second term is due the sphericity. The third term depends of all past history of the process. For the processes with sufficiently large bubbles ($R \sim 1\,\mathrm{mm}$) with moderate change of pressure ($p_e(t)/p_0 \leqslant 10$) the second term is small.

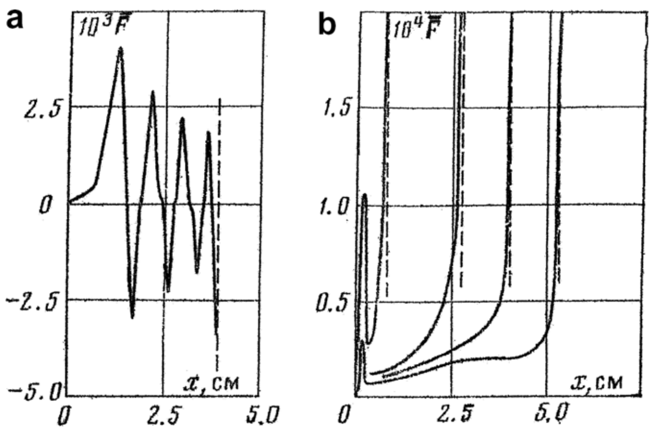

Fig. 2. Integrands in Duhamel integral for the behavior of vapor bubble in steady shock waves with oscillatory (a) and monotonic structure (b).

The first term is zero in the cases when temperature at the interface is changing continuously starting from T_0.

For this reason in large class of the problems we can write [11].

$$q_e = \frac{\lambda_e}{\sqrt{\pi a_e}} \int_0^t \frac{dT_s(\tau)}{d\tau} \frac{d\tau}{\sqrt{t-\tau}} \tag{4.4}$$

After using (4.2) we will obtain

$$q_e = \frac{\lambda_e}{\sqrt{\pi a_e}} \int_0^t \frac{T_s}{\ell \rho_v} \frac{dp_v}{d\tau} \frac{d\tau}{\sqrt{t-\tau}} \tag{4.5}$$

In Fig. 2 nondimensional integrands in Duhamel integral are presented

$$F(\tau) = \frac{1}{\sqrt{t-\tau}} \frac{dT_s}{d\tau}, t = \int_0^x \frac{dz}{V(z)} \tag{4.6}$$

The cases a and b in Fig. 2 correspond to the behavior of vapor bubble in stationary shock waves with oscillatory (a) and monotonic structure (b) [11]. In Fig. 2b the curves are presented for different moments of time that corresponds to different distances. The dashed lines show the position of the shock wave.

The comparison of formulas (3.13) and (4.5) shows that in both cases (gas and vapor bubbles), the heat fluxes have the similar structure (Duhamel integral with rooted kernel).

5. Conclusion

The solution of heat equation inside oscillating gas bubble with moving boundary was obtained by Fourier's method. The formula for interface heat flux containing Jacobi theta-function was derived. The kernel of the integral is represented by a series of exponential functions. A simple approximation with high accuracy for theta-function obtained earlier, is used for obtaining the expression for the interface heat flux. The asymptotic formula for the heat flux is Duhamel integral with rooted kernel.

In the case of vapor bubbles with phase transition the major role plays external heat problem in liquid. It is shown that the asymptotic expression for the heat flux at the interface for gas bubbles has the similar structure as the heat flux from the vapor bubble surface to the liquid. In both cases it is Duhamel integral with rooted kernel.

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